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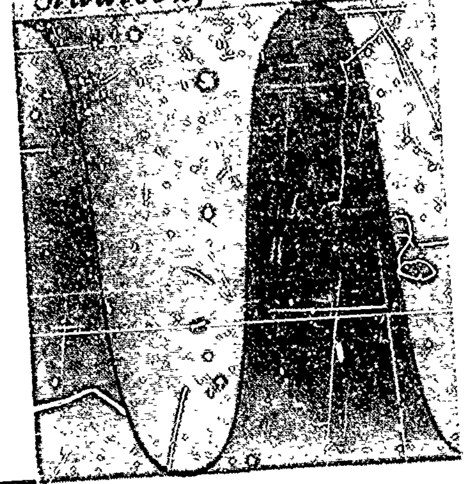

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GENERALIZED STRAIN MEASURE WITH APPLICATIONS TO PHYSICAL PROBLEMS

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1. Introduction

The concept of stress is well defined through the primitive concepts of force and area-vector. Such is not the case with strain for which displacement-gradients have to be used, with the result that we find various measures of it used in literature. In particular we have measures due to Cauchy, Green, Hencky, Almansi, Swainger and Wall. The classical theory of elasticity employs the Cauchy measure when the strain is small and only linear terms in displacement gradients referred to the unstrained state are retained. The theory of finite deformation uses the Almansi or Green measure. Hencky's measure is useful in plasticity. Swainger uses linear displacement gradients but referred to the strained state. Wall combines some of these measures to get new ones. It may therefore be of some interest to suggest a generalized strain measure which includes all of them as particular cases.

In general it is difficult to find stress-strain tensor relations suitable for giving good quantitative results. The generalized strain measure can indicate how a particular result may be extended for obtaining better agreement with

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experimental data. It may be possible to derive these improved results by taking the elastic constants as functions of strain invariants and not as absolute constants. Thus a reasonable justification may be obtained for a number of ad hoc empirical formulae used in literature. We may also find that the recent tendency to use new coefficients of the medium may not be pursued in all cases.

2. Generalized Strain Measure

As the strain tensor $\underline{e}_{ij}$ is symmetric we may associate it with a quadric surface, known as Cauchy's strain quadric, at any point P, such that

$$\underline{e}_{ij}\underline{x}^i\underline{x}^j = \text{a constant} \quad . \tag{2.1}$$

If $\underline{\Lambda}_i$ be its principal axes and $\underline{a}_i$ and $\underline{x}_i$ are the coordinates before and after strain we may write

$$\sum d\underline{a}_i^2 = \sum d\underline{x}_i^2/\underline{\Lambda}_i^2 \; . \tag{2.2}$$

Any suitable function of $\underline{\Lambda}_i$ which vanishes for $\underline{\Lambda}_i = 1$ when there is no deformation may constitute a measure of strain. If we put

$$\underline{\Lambda} = (1 - n\,\underline{e})^{-1/n} \quad , \tag{2.3}$$

we notice that it includes all the known measures. In fact we have the following results:

$$\begin{aligned}
n &= -1, & \underline{\Lambda} &= 1 + \underline{e}\,, & &\text{Cauchy } (\underline{\underline{C}}) \\
n &= -2, & \underline{\Lambda} &= (1 + 2\,\underline{e})^{\frac{1}{2}}, & &\text{Green } (\underline{\underline{G}}) \\
n &= 0, & \underline{\Lambda} &= \exp(\underline{e})\,, & &\text{Hencky } (\underline{\underline{H}}) \\
n &= 2, & \underline{\Lambda} &= (1 - 2\underline{e})^{-\frac{1}{2}}, & &\text{Almansi } (\underline{\underline{A}}) \\
n &= 1, & \underline{\Lambda} &= (1 - e)^{-1}, & &\text{Swainger } (\underline{\underline{S}}) \\
n &= \infty, & \underline{\Lambda} &= 1\,, & &\text{No strain } (\underline{\underline{N}})
\end{aligned} \tag{2.4}$$

We can therefore call n the coefficient of strain measure and $\underline{\underline{A}}$ as a generalized strain measure. The strain components of the measures in (2.4) have been calculated by Cauchy, Green (1), Murnaghan (2), Reiner and Hanim (3) and others. Seth (4) has stressed the use of the $\underline{\underline{A}}$ - measure in a number of applications.

We shall now illustrate the use of the generalized strain measure by a few examples. In the case of simple shear it will be shown that Rivlin's (5) result that no two of the normal stresses may be equal can be obtained without using any additional elastic constant.

3. Homogeneous Pure Strain

In this case the displacement is given by

$$\underline{\underline{u}} = \underline{\underline{c}}\,\underline{\underline{x}} \;, \quad \underline{\underline{c}} = \begin{bmatrix} \underline{c}_1 & 0 & 0 \\ 0 & \underline{c}_2 & 0 \\ 0 & 0 & \underline{c}_3 \end{bmatrix} \tag{3.1}$$

The principal pure strains are:

$$\underline{e}_{11} = \frac{1}{n}(1 - \underline{c}_1^{-n}), \quad \underline{e}_{22} = \frac{1}{n}(1 - \underline{c}_2^{-n}), \quad \underline{e}_{33} = \frac{1}{n}(1 - \underline{c}_3^{-n}) \;. \tag{3.2}$$

The strain invariants are:

$$I = \frac{1}{n}(3 - \sum \underline{c}_i^{-n}) \;,$$

$$II = \frac{1}{2n^2}\sum (1 - \underline{c}_i^{-n})(1 - \underline{c}_j^{-n}) \;, \quad i \neq j \;, \quad i, j = 1, 2, 3, \tag{3.3}$$

$$III = \frac{1}{n^3}(1 - \underline{c}_1^{-n})(1 - \underline{c}_2^{-n})(1 - \underline{c}_3^{-n}) \;.$$

The cubical dilatation $\epsilon_v = (v/v_0 - 1)$, is

$$(1 - nI + n^2 II - n^3 III)^{-1/n} - 1 \;. \tag{3.4}$$

For $n = 0$, which is $\underline{\underline{H}}$ - measure, it takes the simpler form $\exp(I) - 1$. For $\underline{\underline{N}}$ - measure it is obviously zero, as it should be.

We now treat the cases of simple tension, hydrostatic pressure, simple shear and yield condition. In each case the results given by using generalized strain and a linear stress-strain tensor relation involving two elastic constants are extended and then justified by taking the elastic constants as functions of the strain invariants.

4. Generalized Tension-Stretch Law

In this case $\underline{c}_1 = \underline{c}_2$ and from (3.1) and (3.2) we get

$$I = \frac{1}{n}(3 - 2\underline{c}_1^{-n} - \underline{c}_3^{-n})$$

$$II = \frac{1}{n^2}(1 - \underline{c}_1^{-n})(3 - c_1^{-n} - 2\underline{c}_3^{-n}) \ , \qquad (4.1)$$

$$1 - \underline{c}_3^{-n} = \frac{1}{3}n[I \pm 2(I^2 - 3II)^{\frac{1}{2}}] \ .$$

The tension-stretch law is found to be governed by the equation[4]

$$T = \frac{1}{n}E[1 - (1 + \underline{s})^{-n}] \ , \qquad (4.2)$$

when E is Young's modulus and s is the ordinary stretch of the classical theory.

For $\underline{\underline{A}}$ - measure $n = 2$ and we get the result due to Seth[4]. From (4.2) we derive the following conclusion, not given by the classical theory.

(i) A yield point is indicated at $T = E/n$.

(ii) Tension and compression results are not similar.

(iii) For infinite compression $\underline{s} \to -1$, $T \to -\infty$, thus showing that no amount of stress can reduce the length to zero.

(iv) $$\frac{dT}{d\underline{s}} = \frac{E}{(1+\underline{s})^{n+1}}$$

This is equal to E only at $\underline{s} = 0$; otherwise the strain increases much faster than stress in tension.

From (4.1) we see that the yield point occurs at $T = \frac{1}{n} E$, which is a very high value if the classical value of E is used. It should therefore be generalized. An obvious extension of it is

$$T = A[1 - \frac{1}{(1+\underline{s})^{\alpha}}] , \tag{4.3}$$

where A and α are constants to be adjusted. A can now represent the yield stress $\mathcal{S}$ and α can be taken as $E/\mathcal{S}$.

Comparing (4.2) with (4.3) we see that we can obtain this later form if we take E as given by

$$\begin{aligned} E &= nA \frac{1 - \underline{c}_3^{-\alpha}}{1 - \underline{c}_3^{-n}} \\ &= \alpha A[1 - \frac{1}{6}(\alpha - n)\{ I \pm 2(I^2 - 3II)^{\frac{1}{2}}\} + \ldots] . \end{aligned} \tag{4.4}$$

As a first approximation $E = \alpha A$; in general it can be taken as a function of the strain invariants I and II.

5. Hydrostatic Pressure

In this case all the $\underline{c}$'s are equal and the pressure-volume relation is found to be

$$p = \frac{3k}{n}[(\frac{v_0}{v})^{\frac{1}{3}n} - 1] . \tag{5.1}$$

For $\underline{\underline{A}}$ - measure it reduces to Murnaghan's[2] result, which gives very good agreement with Bridgeman's[6] result on the compression of sodium at high pressures.

Here $\underline{k}$ is the ordinary coefficient of cubical expansion. To get better agreement with results at very high pressures, which may be of the order of 20,000 atmospheres, Murnaghan, without giving any justification, generalized (5.1) into

$$\underline{p} = A\,[(\frac{v_0}{v})^{\alpha} - 1]\ , \tag{5.2}$$

and determined A and α from Bridgeman's experimental data. We can readily see that (5.2) can be derived from (5.1) if $\underline{k}$ is taken as a suitable function of the invariant I. From (3.2) we have

$$I = \frac{3}{n}(1 - \underline{c}_1^{-n})\ ,\ \underline{c}_1^3 = v/v_0\ .$$

Hence

$$\begin{aligned} \underline{k} &= \frac{1}{3} n A \frac{(v_0/v)^{\alpha} - 1}{(v_0/v)^{\frac{1}{3}n} - 1} \\ &= \frac{A}{I}[1 - (1 - \frac{1}{3} n\, I)^{3\alpha/n}] \qquad (5.3) \\ &= A\alpha\,[1 - \frac{1}{6}(3\alpha - n)I + \dots]\ , \end{aligned}$$

which is a type of expansion sometimes used for $\underline{k}$ and which is similar to (4.4). In general this expansion is not valid, but $\underline{k}$ remains a function of I.

6. Simple Shear

In this case $\underline{\underline{c}}$ is given by

$$\underline{\underline{c}} = \begin{bmatrix} 1 & \underline{c} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ .$$

Following Love[7] and Reiner[3] we find the following values of the principal strains:

$$\underline{e}_{11}, \underline{e}_{22} = \frac{1}{n}[1 - 2^n\{(4 + \underline{c})^{\frac{1}{2}} \pm c\}^{-n}] , \tag{6.1}$$

$$\underline{e}_{33} = 0 .$$

The generalized strain components are then given by

$$\begin{aligned} \underline{e}_{xx}, \underline{e}_{yy} &= \frac{1}{2}(\underline{e}_{11} + \underline{e}_{22}) \pm \frac{1}{2}c(4 + \underline{c}^2)^{-\frac{1}{2}}(e_{11} - e_{22}) , \\ \underline{e}_{xy} &= (4 + \underline{c}^2)^{-\frac{1}{2}}(\underline{e}_{11} - \underline{e}_{22}), \quad \underline{e}_{zz}, \underline{e}_{zx}, \underline{e}_{yz} = 0 . \end{aligned} \tag{6.2}$$

For $\underline{\underline{A}}$-measure these reduce to $\underline{e}_{xx} = 0$, $\underline{e}_{yy} = -\frac{1}{2}\underline{c}^2$, $\underline{e}_{xy} = \frac{1}{2}\underline{c}$; for $\underline{\underline{G}}$-measure they become $\underline{e}_{xx} = 0$, $\underline{e}_{yy} = \frac{1}{2}\underline{c}^2$, $\underline{e}_{xy} = -\frac{1}{2}\underline{c}$. Inall other measures $\underline{e}_{xx} \neq 0$.

The stress components are given by

$$\begin{aligned} \underline{\tau}_{xx} &= \Lambda(\underline{e}_{xx} + \underline{e}_{yy}) + 2\mu\,\underline{e}_{xx} \\ \underline{\tau}_{yy} &= \Lambda(\underline{e}_{xx} + \underline{e}_{yy}) + 2\mu\,\underline{e}_{yy} \\ \underline{\tau}_{zz} &= \Lambda(\underline{e}_{xx} + \underline{e}_{yy}) + 2\mu\,\underline{e}_{zz} \\ \underline{\tau}_{xy} &= 2\mu\,\underline{e}_{xy} , \quad \underline{\tau}_{yz}, \underline{\tau}_{zx} = 0 . \end{aligned} \tag{6.3}$$

Thus we get normal stresses which give rise to Kelvin and Poynting effects.[1] In both cases such stresses have to be applied to keep the body in equilibrium. As $\underline{e}_{xx}$ is zero only in $\underline{\underline{A}}$ - and $\underline{\underline{G}}$ - measures, the normal stresses $\underline{\tau}_{xx}$ and $\underline{\tau}_{zz}$ are generally unequal, as pointed out by Rivlin[5]. We obtain this result without using any additional elastic coefficient. From (6.3) we see that the normal stresses can be taken in the form

$$\underline{\tau}_{11} = A_{11}\underline{e}_{11} + B_{11}\underline{e}_{22}$$
$$= \frac{A_{11}}{n}\left[1 - 2^n \left\{(4 + \underline{c}^2)^{\frac{1}{2}} + \underline{c}\right\}^{-n}\right] \tag{6.3}$$
$$+ \frac{B_{11}}{n}\left[1 - 2^n \left\{(4 + \underline{c}^2)^{\frac{1}{2}} - c\right\}^{-n}\right]$$

For $\underline{\underline{A}}$ - and $\underline{\underline{G}}$ - measures we get $\underline{\tau}_{11}$ proportional to $\underline{c}^2$ as has been found by Rivlin [5], Truesdell [1], Green [8] and others.

The shearing stress $\underline{\tau}_{xy}$ is given by

$$\underline{\tau}_{xy} = \frac{2^n}{n} \frac{\left[(4+\underline{c}^2)^{\frac{1}{2}} - \underline{c}\right]^{-n} - \left[(4+\underline{c}^2)^{\frac{1}{2}} + \underline{c}\right]^{-n}}{(4 + c^2)^{\frac{1}{2}}} \tag{6.4}$$

For A - measure it takes the simple form $\frac{1}{2}c$.

The results in (6.3) and (6.4) may be generalized by replacing $-n$ with α and a justification on the lines given in Sections 5 and 6 may be given.

7. Generalized Strain in Hollow Spheres and Cylinders

For the symmetrical deformation of a thick spherical shell subjected to uniform internal and external pressures, the radial displacement, strain components and the stress components are given by [4]

$$\underline{u}_r = r(1 - P)$$
$$\underline{e}_{rr} = \frac{1}{n}\left[1 - P^n\{1 + v(v + 2)\}^{\frac{1}{2}n}\right] , \tag{7.1}$$

$$\underline{e}_{\varphi\varphi} = \underline{e}_{\phi\phi} = \frac{1}{n}(1 - P^n) \ , \tag{7.2}$$

$$\underline{e}_{r\varphi} \ , \ \underline{e}_{\varphi\phi} \ , \ \underline{e}_{\phi r} = 0 \ ,$$

$$\underline{\tau}_{rr} = \frac{1}{n}\Lambda[3 - 2P^n - P^n f(v)] + \frac{2\mu}{n}[1 - P^n f(v)]$$

$$\underline{\tau}_{\varphi\varphi} = \underline{\tau}_{\phi\phi} = \frac{1}{n}\Lambda[3 - 2P^n - P^n f(v)] + \frac{2\mu}{n}[1 - P^n]$$

$$\underline{\tau}_{r\varphi} \ , \ \underline{\tau}_{\varphi\phi} \ , \ \underline{\tau}_{r\phi} = 0 \ , \tag{7.3}$$

where $V = \frac{r}{P}\frac{dP}{dr}$, $f(v) = [1 + v(v+2)]^{\frac{1}{2}n}$. P may be determined from a non-linear differential equation resulting from the body-stress equations. At present we shall obtain a yield condition from (7.3) and show that it holds good in all strain measures.

From (7.3) we have, on eliminating P^n

$$\frac{n\underline{\tau}_{rr} - \underline{\tau}_{\varphi\varphi}}{3\Lambda + 2\mu - n\tau_{rr}} = \frac{2\mu[1 - f(v)]}{2\Lambda + 2f(v) + 2\mu f(v)} \tag{7.4}$$

If r' is the radius vector before strain, we have

$$r' = rP \ ,$$

$$\text{and} \quad \frac{dr'}{dr} = P(1 + v) \ . \tag{7.5}$$

These show that $v \to -1$ for infinite extension and $v \to \infty$ for infinite compress

From (7.4) we thus get the inequality

$$-\frac{1 - 2\sigma}{1 - \sigma} < \frac{n(\underline{\tau}_{rr} - \underline{\tau}_{\varphi\varphi})}{3\Lambda + 2\mu - n\underline{\tau}_{rr}} < \frac{1 - 2\sigma}{2\sigma} \ . \tag{7.6}$$

Thus, whatever the generalized strain may be and whatever values Λ and

(constant or variable) may have the yield condition in extension or compression may be written as

$$(\underline{\tau}_{rr} - \underline{\tau}_{\varphi\varphi}) + A_1(\underline{\tau}_{rr} - \underline{\tau}_{\phi\phi}) + A_2(\underline{\tau}_{rr} + \underline{\tau}_{\varphi\varphi} + \underline{\tau}_{\phi\phi}) = k_1 \tag{7.7}$$

when A_1 and A_2 are independent of n.

Assuming that

$$(\underline{\tau}_{rr} - \underline{\tau}_{\varphi\varphi}) > \underline{\tau}_{rr} - \tau_{\phi\phi} \quad ,$$

we may write the generalized yield condition in the form

$$\underline{\tau}_{11} - \underline{\tau}_{33} + F_1(\mathrm{I, II, III})(\underline{\tau}_{11} + \underline{\tau}_{22} + \underline{\tau}_{33}) = F_2(\mathrm{I, II, III}) \quad , \tag{7.8}$$

F_1 and F_2 being functions of the invariants I, II, and II, and $\underline{\tau}_{11}$, $\underline{\tau}_{22}$, $\underline{\tau}_{33}$ the principal stresses in descending order of magnitude.

The yield condition in (7.8) is true for isotropic and aelotropic materials and for all measures of strain.

If the mean stress is negligible we get

$$\underline{\tau}_{11} - \underline{\tau}_{33} = F_2 \quad , \tag{7.9}$$

which is a generalized form of Tresca's yield condition.

Similar results may be obtained from the case of hollow cylinders.

Summary -

All the known strain measures can be represented by the generalized strain measure $(1 - n\underline{e})^{-1/n}$ where n may be called the coefficient of strain measure. Any stress-strain tensor relation cannot give a complete quantitative picture. The result obtained by using any one of them will have to be generalized to conform to experimental data. The generalization may be justified by assuming the coefficients of the medium to be functions of the strain invariants. The introduction of new coefficients of the medium may therefore be not found necessary in many cases.

REFERENCES

1. Exhaustive references to the works of Cauchy, Green, Almansi and others will be found in Truesdell, C. "The mechanical foundations of elasticity and fluid mechanics", J. Rational Mech. Anal., 1 (1952), 125-320; 2(1953), 593-616 and subsequent papers.

2. Murraghan, F. D., "Finite deformations of an elastic solid", Amer. J. Math., 59(1937), 253-260.

   "The compressibility of solids under extreme pressures", Karman anniv. Vol., 1941, 121-136.

   "The compressibility of media under extreme pressures", Proc. Nat. Acad. Sci. U.S.A. 30(1944), 244-247.

3. Reiner, M., "Deformation and flow of solids", Springer Verlag, 1956, 203-207. Karni, Z. and Reiner, M., "Measures of deformation in the strained and unstrained state", Bull. Res. Course, Isreal, 6C (1958), 113.

4. Seth, B. R., "Finite strain in elastic problems", Phil. Trans. Roy. Soc. London (A), 234 (1935), 231-264.

   and Shepherd, W. M., "Fine strain in elastic problems, II", Proc. Roy. Soc. London (A), 156(1936), 171-192.

   "Some recent applications of the theory of finite-deformation", Sym. Amer. Math. Soc., 3(1950), 67-94.

   "Non-linear continuum mechanics", Presidential address, (Section of Mathematics). Proc. 42nd Ind. Sci. Congress, II (1955).

   "Finite bending of a plate into a spherical shell", Zeit. Angew. Math. Mech. 37(1957), 393-398.

"Poro-plastic deformation", Rheo. Acta. 2/3 (1958), 316-318.

"Finite strain in engineering design", Proc. Inter. Congress on Math. for the engineer, Brusells, 1958, 386-390.

"Non-homogeneous yield condition", Proc. IUTAM Sym. on non-homogenity in elasticity and plasticity", Warsaw, 1959.

"Finite deformation of cylindrical shells", Proc. IUTAM Sym. on theory of thin elastic shells, Delfi, 1959, 355-363.

"Finite bending of plates into cylindrical shells", Signorini memorial volume, Ann. Mat. Bologne, 50 (1960), 119-125.

"Finite deformation of plates into shells", Proc. Inter. Conf. on partial diff. equ. and continuum mechanics, Madison, 1961, 95-105.

"Stability of finite deformation", Problems of continuum mechanics, Muskhelishvili memorial volume, Soc. Indust. Appl. Math., 1961. 406-414; USSR Acad. Sci., Moscow, 1961, 359-365.

5. Rivlin, R. S. "Large elastic deformations of isotropic materials, I-VIII", Phil. Trans. Roy. Soc. London (A), 240(1948), 459-490; 491-508; 509-525; 241(1948), 379-397 and subsequent papers.

"Solution of problems in second order elasticity theory", Jour. Rat. Mech. Anal. 2 (1953), 53-81.

and Erickson, J. L., "Large elastic deformations of homogeneous anistropic materials", Jour. Rat. Mech. Anal. 3(1954), 281-301.

6. Bridgeman, P. W., "Electrical resistences and volume changes up to 20,000 Kq/cm^2," Proc. Nat. Acad. Sci., U.S.A., 21(1935), 109.

7. Love, A. E. H., "Mathematical theory of elasticity", 4th ed., Cambridge (1934).

8. Green, A. E. and Zerna, W., "Theoretical elasticity", Oxford, (1954).

Green, A. E., and Adkins, J. E., "Large elastic deformations", Oxford, (1960).